Positive and completely positive maps via free additive powers of probability measures

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  4. Fast simulation of quantum systems

Entanglement in Quantum Information Theory

- Quantum states with n degrees of freedom are described by density matrices

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- For rank one quantum states, entanglement can be detected and quantified by the entropy of entanglement

$$E_{\mathrm{ent}}(P_x) = H(s(x)) = -\sum_{i=1}^{\min(m,n)} s_i(x) \log s_i(x),$$

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- A pure state $x \in \mathbb{C}^m \otimes \mathbb{C}^n$ is separable $\iff$ $E_{\mathrm{ent}}(P_x) = 0$.

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- Let $f : \mathbb{M}_n \to \mathcal{A}$ be a positive map. Then, for every separable state $\rho_{12} \in \mathbb{M}_{mn}^{1,+}$, one has $[\mathrm{id}_m \otimes f](\rho_{12}) \geq 0$.
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- Hence, positive, but not CP maps f provide sufficient entanglement criteria: if $[\mathrm{id}_m \otimes f](\rho_{12}) \not\geq 0$, then ρ_{12} is entangled.
- Moreover, if $[\mathrm{id}_m \otimes f](\rho_{12}) \geq 0$ for all positive, but not CP maps $f : \mathbb{M}_n \to \mathbb{M}_m$, then ρ_{12} is separable.

Positive Partial Transpose matrices

- The transposition map $\mathrm{t} : A \mapsto A^t$ is positive, but not CP. Define the convex set

$$\mathcal{PPT} = \{\rho_{12} \in \mathbb{M}_{mn}^{1,+} \mid [\mathrm{id}_m \otimes \mathrm{t}_n](\rho_{12}) \geq 0\}.$$

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- Low dimensions are special because every positive map $f : \mathbb{M}_2 \to \mathbb{M}_{2/3}$ is decomposable:

$$f = g_1 + g_2 \circ \mathrm{t},$$

with $g_{1,2}$ completely positive. Among all decomposable maps, the transposition criterion is the strongest.

The PPT criterion at work

- Recall the Bell state $\rho_{12} = P_{\mathrm{Bell}}$, where

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- Written as a matrix in $\mathbb{M}^{1,+}_{2\cdot 2}$

$$\rho_{12} = \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right) = \frac{1}{2}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

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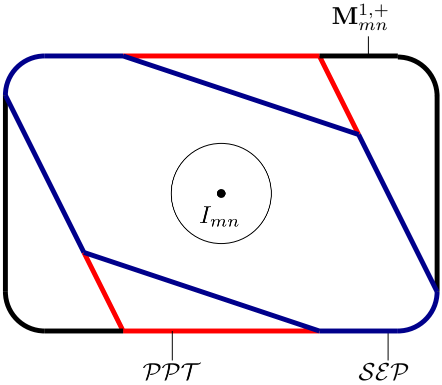
- This matrix is no longer positive $\implies$ the state is entangled.

Three convex sets



The inclusions $\mathcal{SEP} \subset \mathcal{PPT} \subset \mathbb{M}_{mn}^{1,+}$ are generally strict.

Our work: two directions

- How good is the PPT criterion ? In other words, how far is the inclusion $\mathcal{SEP} \subset \mathcal{PPT}$ from being an equality ?
  - Compare volumes of sets $\rightsquigarrow$ random quantum states.
  - Threshold phenomenon: depending on the probability used to measure volumes, either almost all states or almost none are PPT (joint work with Aubrun, Banica).
  - Same phenomenon observed for $\mathcal{SEP}$ by Aubrun, Szarek and Ye.
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- How to produce other maps which are positive but not CP and lead to interesting separability criteria ?

The Choi matrix of a map

- For any n, recall that the maximally entangled state is the orthogonal projection onto

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Theorem (Choi '72)

A map $f : \mathbb{M}_n \to \mathcal{A}$ is CP iff its Choi matrix C_f is positive.

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- The map $f \mapsto C_f$ is called the Choi-Jamiolkowski isomorphism.
- It sends:
  1. All linear maps to all operators;
  2. Hermicity preserving maps to hermitian operators;
  3. Entanglement breaking maps to separable quantum states;
  4. Unital maps to operators with unit left partial trace ($[\mathrm{Tr} \otimes \mathrm{id}]C_f = \mathrm{I}_d$);
  5. Trace preserving maps to operators with unit left partial trace ($[\mathrm{id} \otimes \mathrm{Tr}]C_f = \mathrm{I}_n$).

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Theorem

A map $f : \mathbb{M}_n \to \mathcal{A}$ is k-positive iff its Choi matrix C_f is k-positive. In particular, f is positive iff C_f is block-positive.

Random Choi matrices

- Let μ be a compactly supported probability measure on $\mathbb{R}$. For each d we introduce a real valued diagonal matrix X_d of $\mathbb{M}_n \otimes \mathbb{M}_d$ whose eigenvalue counting distribution converges to μ and whose extremal eigenvalues converge to the respective extrema of the support of μ.

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- Let U_d be a random Haar unitary matrix in the unitary group $\mathcal{U}_{nd}$, and $f_\mu^{(d)} : \mathbb{M}_n \to \mathbb{M}_d$ be the map whose Choi matrix is $U_d X_d U_d^*$.

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- Let U_d be a random Haar unitary matrix in the unitary group $\mathcal{U}_{nd}$, and $f_\mu^{(d)} : \mathbb{M}_n \to \mathbb{M}_d$ be the map whose Choi matrix is $U_d X_d U_d^*$.

Theorem

Under the above assumptions, if $\mathrm{supp}(\mu^{\boxplus n/k}) \subset (0, \infty)$ *then, almost surely as* $d \to \infty$*, the map* $f_\mu^{(d)}$ *is k-positive.*

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Proof ingredients

Let $f_\mu^{(d)} : \mathbb{M}_n \to \mathbb{M}_d$ be the map whose Choi matrix is $U_d X_d U_d^*$.

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If $\operatorname{supp}(\mu^{\boxplus n/k}) \subset (0, \infty)$ then, almost surely as $d \to \infty$, the map $f_\mu^{(d)}$ is k-positive. If $\operatorname{supp}(\mu^{\boxplus n/k}) \cap (-\infty, 0) \neq \emptyset$ then, almost surely as $d \to \infty$, $f_\mu^{(d)}$ is not k-positive.

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Proposition

A map f is k-positive iff for any self-adjoint projection $P \in \mathbb{M}_n$ of rank k, the operator $P \otimes I_d \cdot C_f \cdot P \otimes I_d$ is positive.

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Proposition (Nica and Speicher)

Let x, p be free elements in a ncps (M, τ) and assume that p is a selfadjoint projection of rank $\tau(p) = 1/t$ $(t \geq 1)$ and that x has distribution μ. Then, the distribution of $t^{-1}pxp$ inside the contracted ncps $(pMp, \tau(p \cdot p))$ is $\mu^{\boxplus t}$

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- Define

$$\begin{aligned} f_\mu : \mathbb{M}_n &\to \mathcal{M} \\ E_{ij} &\mapsto E_{1i}XE_{j1} \end{aligned}$$

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$$f_\mu : \mathbb{M}_n \to \mathcal{M}$$
$$E_{ij} \mapsto E_{1i} X E_{j1}$$

Theorem

The map f_μ is k-positive iff $\mathrm{supp}(\mu^{\boxplus n/k}) \subseteq \mathbb{R}_+$.

Example: semicircular measures

- Let $s_{a,\sigma}$ be the semi-circle distribution of mean a and variance σ^2, having support $[a - 2\sigma, a + 2\sigma]$.

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- We have $s_{a,\sigma}^{\boxplus n/k} = s_{an/k, \sigma\sqrt{n/k}}$, with support $\mathrm{supp}(s_{a,\sigma}^{\boxplus n/k}) = [an/k - 2\sigma\sqrt{n/k}, an/k + 2\sigma\sqrt{n/k}]$.

Theorem

Let n be an integer and a, σ positive parameters. The map $f_{a,\sigma} : \mathbb{M}_n \to \mathcal{M}$ associated to a semi-circular distribution $s_{a,\sigma}$ is k-positive iff $k \leq 4n\sigma/a^2$. In particular, for any n and any $k < n$, there exist parameters $a, \sigma > 0$ such that the above map is k-positive but not $(k+1)$-positive.

The End